Non-linear Programming Problem

— A General non-linear programming problem (GNLPP) is defined as follows:

Determine the values of $x_1, x_2, x_3, \ldots, x_n$ which optimize the function

$$Z = f(x_1, x_2, \ldots x_n)$$

subject to the constraints

$$g_1(x_1, x_2, \ldots, x_n) \leq \text{ or } = \text{ or } \geq b_1$$

$$g_2(x_1, x_2, \ldots, x_n) \leq \text{ or } = \text{ or } \geq b_2$$

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$$g_m(x_1, x_2, \ldots, x_n) \leq \text{ or } = \text{ or } \geq b_m$$

and the non-negative restrictions

$$x_j \geq 0, \; j = 1, 2, 3, \ldots n.$$

where either $f(x_1, x_2, \ldots x_n)$ or some or all $g_i(x_1, x_2, \ldots x_n)$, $i = 1, 2, \ldots m$ are non-linear.

In matrix form the above problem can be stated as

optimize $Z = f(X)$

s.t.

$$g_i(X) \leq \text{ or } = \text{ or } \geq b_i \quad \forall \, i = 1, 2, \ldots m$$

and $X \geq 0$

$$X = (x_1, x_2, x_3, \ldots x_n)^T$$

Solution of NLPP with all equally Constraints. ②

– If the objective function of a NLPP is continuous and differentiable and all the constraints are equality constraint (i.e. equations) then they can be solved by the Lagrangian Multipliers.

Theorem:

The necessary condition for the optimality of the objective function of two decision variables and one equality constraints.

Solution:

Let us consider the non-linear programming problem

$$\text{opt. } Z = f(x_1, x_2)$$

S.t.

$$g(x_1, x_2) = b$$

$$x_1, x_2 \geq 0.$$

Let $h(x_1, x_2) = g(x_1, x_2) - b = 0$

and λ be a constant known as Lagrangian Multipliers and define a function $L(x_1, x_2, \lambda)$ by

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\, h(x_1, x_2) \quad -① $$

called Lagrangian function.

Now, suppose L, f, h are differentiabl partially w.r.t. x_1, x_2 and λ resp. Then the necessary condition for the maxima and minima of $f(x_1, x_2)$ subject to $h(x_1, x_2) = 0$ are given by

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \lambda} = -h(x_1, x_2) = 0.$$

or,

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x_1} &= \lambda \frac{\partial h}{\partial x_1} \\ \frac{\partial f}{\partial x_2} &= \lambda \frac{\partial h}{\partial x_2} \\ h(x_1, x_2) &= 0 \end{aligned}}$$

are the required necessary condn for the Existence of maxima & minima of the objective function $f(x_1, x_2)$

Similarly, the necessary conditions for the maxima and minima of the objective function in n decision variables having only one equality constraint is

Let us consider the NLPP

Opt. $Z = f(x_1, x_2, \ldots x_n)$

s.t.

$$g(x_1, x_2, \ldots, x_n) = b$$

and $x_1, x_2 \ldots x_n \geq 0$

Let $h(x_1, x_2, \ldots x_n) = g(x_1, x_2, \ldots, x_n) - b = 0$

$$L(x_1, x_2, \ldots x_n, \lambda) = f(x_1, x_2, \ldots x_n) - \lambda h(x_1, x_2, \ldots x_n)$$

i.e. $L(X, \lambda) = f(X) - \lambda h(X)$.

$$X = (x_1, x_2, \ldots x_n)^T.$$

Let L, f and h are all partially differentiable w.r.t. $x_1, x_2, \ldots x_n$ and λ.

∴ The necessary conditions for the maximum or minimum of $f(X)$ subject to $h(X) = 0$ are given by

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \quad \forall\ j = 1, 2, \ldots n$$

and $\dfrac{\partial L}{\partial \lambda} = -h(X) = 0$

i.e. $$\boxed{\begin{array}{l} \dfrac{\partial f}{\partial x_j} = \lambda \dfrac{\partial h}{\partial x_j} \quad \forall j = 1, 2, \ldots n \\ \text{and} \quad h(X) = 0 \end{array}}$$

* n decision variables and two equality constraints.

i.e. opt. $Z = f(x_1, x_2, \cdots x_n)$

s.t.

$g_1(x_1, x_2, \cdots x_n) = b_1 \longrightarrow h_1(X) = g_1(X) - b_1 = 0$

$g_2(x_1, x_2, \cdots x_n) = b_2 \rightarrow h_2(X) = g_2(X) - b_2 = 0$

$x_1, x_2, \cdots x_n \geq 0$

$L(x_1, x_2, \cdots x_n; \lambda_1, \lambda_2) = f(x_1, x_2 \cdots x_n) - \lambda_1 h_1(X) - \lambda_2 h_2(X)$

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{x_j} - \lambda_1 \frac{\partial h_1}{\partial x_j} - \lambda_2 \frac{\partial h_2}{\partial x_j} = 0 \quad \forall j = 1, 2, \cdots n$$

$$\frac{\partial L}{\partial \lambda_1} = -h_1(X) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -h_2(X) = 0$$

The necessary cond^n for max. or min. of $f(X)$ s.t. $h_1(X) = g_1(X) - b_1 = 0$ and $h_2(X) = g_2(X) - b_2 = 0$

$X = (x_1, x_2, \cdots x_n)^T$ are.

$$\boxed{\begin{array}{l} \dfrac{\partial f}{\partial x_j} = \lambda_1 \dfrac{\partial h_1}{\partial x_j} + \lambda_2 \dfrac{\partial h_2}{\partial x_j} \quad ; j = 1, 2, \ldots n \\ h_1(X) = 0 \\ h_2(X) = 0 \end{array}}$$

*. n decision variables and m equality constraints $m<n$.

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial x_j} = \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j} \quad \forall\, j=1,2,\dots n \\ &h_i(X) = 0 \quad \forall\, i=1,2,\dots m(<n) \end{aligned}}$$

Sufficient Cond^ns for maximum or minimum of the objective function

— The necessary cond^ns for maximum or minimum of the objective function in NLPP with equality constraints also become the sufficient conditions for maximum of the objective function if it is concave and for a minimum of the objective function if it is convex.

Let us denote

$$U = \left[\frac{\partial h_i(X)}{\partial x_j}\right]_{m\times n} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

$$V = \left[\frac{\partial^2 L(X,\lambda)}{\partial x_i x_j}\right]_{n\times n} = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

O is an $m\times m$ null matrix.

The square matrix H^B of order $(m+n)\times(m+n)$ called Bordered Hessian matrix defined as follows

$$H^B = \left[\begin{array}{c|c} O & U \\ \hline U^T & V \end{array}\right]_{(m+n)\times(m+n)}$$

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If (X_0, λ_0) be a stationary point for the function $L(X_0, \lambda)$ and H_0^B the value of the corresponding bordered Hessian matrix H^B at this stationary point Then

1) The point X_0 gives maximum value of the objective function if, starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B are of alternate signs, starting with $(-1)^{m+n}$ sign.

2) The point X_0 gives the minimum value of the objective function, starting with the principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B are of the sign $(-1)^m$.

(Page 830, R.K. Gupta).